Universal and nonuniversal tails of distribution functions in the directed polymer and Kardar-Parisi-Zhang problems

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The optimal-fluctuation approach is applied to study the most distant (nonuniversal) tails of the free-energy distribution function $P_L(F)$ for an elastic string (of a large but finite length L) interacting with a quenched random potential. A further modification of this approach is proposed which takes into account the renormalization effects and allows one to study the closest (universal) parts of the tails. The problem is analyzed for different dimensions of a space in which the polymer is imbedded. In terms of the stochastic growth problem, the same distribution function describes the distribution of heights in the regime of a nonstationary growth in the situation when an interface starts to grow from a flat configuration.

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I. INTRODUCTION

A large variety of physical systems can be described in terms of an elastic string interacting with a quenched random potential. The role of such a string can be played by a domain wall in a two-dimensional magnet, a vortex line in a superconductor, a dislocation in a crystal, and so on; however following Ref. 1 the systems of such a kind are usually discussed under the generic name of a directed polymer in a random medium. The unfading interest to this problem is additionally supported by its resemblance to more complex systems with quenched disorder (e.g., spin glasses), as well as by its close relation to the dynamics of a randomly stirred fluid and to the problem of a stochastic growth (see Refs. 2 and 3 for reviews).

One of the main objects of interest in the directed polymer problem is $P_L(F)$, the free-energy distribution function for large polymer length L. In particular, the knowledge of this distribution function allows one to make conclusions on the distribution of displacements. The first important step in the analysis of $P_L(F)$ was made twenty years ago by Kardar,[4] who suggested that all moments of $P_L(F)$ can be found by calculating the moments $Z_n \equiv \overline{Z^n}$ of the distribution of the partition function Z and proposed an asymptotically exact method for the calculation of Z_n in a $(1+1)$-dimensional system (a string confined to a plane) with a δ-correlated random potential. However, soon after that Medina and Kardar[5] understood (see also Ref. 6) that the information provided by the approach introduced in Ref. 4 is insufficient for finding any of the moments of $P_L(F)$. However, it allows one to find[7] the tail of $P_L(F)$ at large negative F (the left tail). In such a situation the conclusions on the width of the distribution function have to rely on the assumption that at large L it acquires a universal form,

$$P_L(F) = \frac{P_*(F/F_*)}{F_*}, \tag{1}$$

incorporating the dependence on all parameters through a single characteristic free-energy scale $F_*(L) \propto L^{\omega}$, which therefore can be extracted from the known form of the tail. The form of Eq. (1) assumes that F, the free energy of a directed polymer in a given realization of a disorder, is counted off from its average, $\bar{F}(L)$ or, more precisely, from the linear in L contribution to F (that is, $L \lim_{L\to\infty}[\bar{F}(L)/L]$). The same convention is implied below.

Only recently it has been understood[8] that the form of the tail following from Zhang's analysis[7] is applicable for the description only of the most distant part of the left tail and therefore has no direct relation to the universal form of the distribution function which is achieved in the limit of $L \to \infty$. At large but finite L the form of $P_L(F)$ given by Eq. (1) can be expected to be achieved only for not too large fluctuations of F [that is, for $|F| \ll F_c(L)$ with $F_c(L)/F_*(L)$ tending to infinity with the increase of L], whereas the behavior of $P_L(F)$ at $|F| \gg F_c(L)$ remains nonuniversal and is not obliged to have anything in common with $P_*(F/F_*)$. In particular, it can incorporate quite different characteristic free-energy scales. Thus, the fact that Zhang's approach[7] reproduces both the correct form of the left tail of $P_*(F/F_*)$ and the correct estimate of the universal free-energy scale $F_*(L)$ (which is the only relevant free-energy scale inside the universal region) is not more than a happy coincidence. In contrast to that, the behavior of the right tail of $P_L(F)$ inside the universal region is qualitatively different from its behavior in the nonuniversal part of the tail.[8]

In this paper, the analysis of the universal and nonuniversal tails of $P_L(F)$ developed in Ref. 8 is presented in more detail and also is extended to the investigation of $(1+d)$-dimensional systems, in which polymer's displacement can be treated as a d-dimensional vector. The paper is organized as follows. In Sec. II we formulate the continuous model which is traditionally applied for the description of the directed polymer problem and briefly review its relation to the Kardar-Parisi-Zhang (KPZ) model[9] of a stochastic growth, as well as to the Burgers turbulence problem. Section III provides a short introduction to the optimal-fluctuation approach, which can be used for the description of the most distant (nonuniversal) parts of the tails of $P_L(F)$. In Ref. 10 an analogous approach has been used to investigate the distribution of velocity and its derivatives in the Burgers turbulence problem, which however required the authors to consider optimal fluctuations with completely different structures then studied here. In Sec. IV the optimal fluctuation approach is applied for the analysis of the far-left tail

of $P_L(F)$, and in Sec. V of the far-right tail. Our main attention is focused on the systems with a δ-correlated random potential; however for $d \geq 2$ the problem with purely δ-functional correlations becomes ill-defined, so we also consider the case when a random potential correlations can be characterized by a finite correlation radius.

For finding the universal parts of both tails, one also has to look for optimal fluctuations, but taking into account that in this regime the parameters of the system have to be considered as scale dependent due to their renormalization by fluctuations. This is done in Sec. VI. The validity of this approach is confirmed by the consistency of its predictions with the results of the exact solution[11] of the (1+1)-dimensional polynuclear growth (PNG) model, as well as by obtaining identical estimates for $F_*(L)$ in the left and right tails. Section VII is devoted to summarizing the results and comparing them with some results of other authors, whereas in Appendix A we discuss how some of the results of this work can be derived in terms of the Kardar-Zhang replica approach.[4,7]

Our main attention throughout this work is focused on a system with free initial condition, that is, we assume that only one end of a string is fixed, whereas the other one is free to fluctuate. In terms of the KPZ problem[9] the same distribution function describes the distribution of heights in the regime of a nonstationary growth in the situation when an interface starts to grow from a flat configuration (L being the total time of the growth). One only has to bear in mind that the height (as defined in the standard form of the KPZ equation) and the free energy of the directed polymer problem differ from each other by the sign. Therefore, what we call here the left (right) tail of $P_L(F)$ in terms of the KPZ problem corresponds to the right (left) tail of the height distribution function. Finally, Appendix B is devoted to demonstrating that when both end points of a directed polymer are fixed, the form of the left tail of $P_L(F)$ remains basically the same as for free initial condition.

II. MODEL

We consider an elastic string in a $(1+d)$-dimensional space interacting with a random potential $V(t,\mathbf{x})$. The coordinate along the average direction of the string is denoted t for the reasons which will become evident few lines below. Such a string can be described by the Hamiltonian,

$$H = \int_0^t dt' \left\{ \frac{J}{2} \left[\frac{d\mathbf{x}(t')}{dt'} \right]^2 + V[t', \mathbf{x}(t')] \right\}, \tag{2}$$

where the first term describes the elastic energy and the second one the interaction with a random potential. Note that the form of the first term in Eq. (2) relies on the smallness of the angle between the string and its preferred direction.

The partition function of a string which starts at $t=0$ and ends at the point $(t,\mathbf{x})$ is then given by the functional integral,

$$z(t,\mathbf{x}) = \int_{-\infty}^{+\infty} d\mathbf{x}' z(0,\mathbf{x}') \int_{\mathbf{x}(0)=\mathbf{x}'}^{\mathbf{x}(t)=\mathbf{x}} \mathcal{D}\mathbf{x}(t') \exp(-H/T), \tag{3}$$

where T is the temperature. Naturally, $z(t,\mathbf{x})$ depends on the initial condition at $t=0$. The fixed initial condition, $\mathbf{x}(t=0)=\mathbf{x}_0$, corresponds to $z(0,\mathbf{x})=\delta(\mathbf{x}-\mathbf{x}_0)$, whereas the free initial condition (which implies the absence of any restrictions on $\mathbf{x}$ at $t=0$) to

$$z(0,\mathbf{x}) = \text{const}. \tag{4}$$

Since Eq. (3) has exactly the same form as the Euclidian functional integral describing the motion of a quantum particle whose mass is given by J in a time-dependent random potential $V(t,\mathbf{x})$ (with t playing the role of imaginary time and T of Plank's constant $\hbar$), the evolution of $z(t,\mathbf{x})$ with the increase in t has to be governed by the imaginary-time Schrödinger equation,

$$-T\frac{\partial z}{\partial t} = \left[-\frac{T^2}{2J}\nabla^2 + V(t,\mathbf{x}) \right] z(t,\mathbf{x}). \tag{5}$$

As a consequence of this, the evolution of the free energy corresponding to $z(t,\mathbf{x})$,

$$f(t,\mathbf{x}) = -T \ln[z(t,\mathbf{x})], \tag{6}$$

is governed[12] by the KPZ equation,[9]

$$\frac{\partial f}{\partial t} + \frac{1}{2J}(\nabla f)^2 - \nu\nabla^2 f = V(t,\mathbf{x}), \tag{7}$$

with the inverted sign of f, where t plays the role of time and $\nu \equiv T/2J$ of viscosity. On the other hand, the derivation of Eq. (7) with respect to $\mathbf{x}$ allows one to establish the equivalence[12] between the directed polymer problem and the Burgers equation[13] with random potential force,

$$\frac{\partial u_a}{\partial t} + u_b \nabla_a u_b - \nu\nabla^2 u_a = \frac{1}{J}\nabla_a V(t,\mathbf{x}), \tag{8}$$

where the vector

$$u_a(t,\mathbf{x}) \equiv \frac{1}{J}\nabla_a f(t,\mathbf{x}) \tag{9}$$

plays the role of velocity. Note that in terms of the KPZ problem the free initial condition (4) corresponds to starting the growth from a flat interface, $f(0,\mathbf{x})=\text{const}$, and in terms of the Burgers problem to starting the evolution from a liquid at rest, $\mathbf{u}(0,\mathbf{x})=0$.

To simplify an analytical treatment, the statistic of a random potential is usually assumed to be Gaussian with

$$\overline{V(t,\mathbf{x})} = 0, \quad \overline{V(t,\mathbf{x})V(t',\mathbf{x}')} = \delta(t-t')U(\mathbf{x}-\mathbf{x}'), \tag{10}$$

where an overbar denotes the average with respect to disorder. Our main attention below is focused on the case of purely δ-functional correlations, $U(\mathbf{x})=U_0\delta(\mathbf{x})$. However, for $d \geq 2$ the problem with such a form of correlations is ill-defined and needs a regularization, so we also consider the case when $U(\mathbf{x})$ can be characterized by a finite correlation radius ξ. On the other hand, we always assume that the correlations in the t direction are δ-functional because in almost

all situations considered below the finiteness of the correlation radius in the t direction can be ignored as soon as it is small in comparison with the total length of a string.

III. OPTIMAL-FLUCTUATION APPROACH

When the distribution of $V(t,\mathbf{x})$ is Gaussian and satisfies Eqs. (10), the probability of any realization of $V(t,\mathbf{x})$ is proportional to $\exp[-S\{V\}]$, where the action $S\{V\}$ is given by the functional

$$S\{V\} = \frac{1}{2}\int_0^L dt \int\int d\mathbf{x}d\mathbf{x}' V(t,\mathbf{x})U^{-1}(\mathbf{x}-\mathbf{x}')V(t,\mathbf{x}'). \tag{11}$$

Here $U^{-1}(\mathbf{x})$ denotes the function whose convolution with $U(\mathbf{x})$ is equal to $\delta(\mathbf{x})$. Accordingly, the probability of any time evolution of $f(t,\mathbf{x})$ is determined by the action $S\{f\}$, which is obtained by replacing $V(t,\mathbf{x})$ in Eq. (11) by the left-hand side of the KPZ equation (7).

To find the most optimal fluctuation having the largest probability (in literature it is often called "instanton"), one has to minimize $S\{f\}$ for the given boundary conditions at $t=0$ and $t=L$. A convenient way to perform such a minimization consists in replacing $S\{f\}$ by

$$S\{f,\mu\} = \int_0^L dt \left\{ \int d\mathbf{x} \left[\frac{\partial f}{\partial t} + \frac{1}{2J}(\nabla f)^2 - \nu\nabla^2 f \right] \mu(t,\mathbf{x}) - \frac{1}{2}\int\int d\mathbf{x}d\mathbf{x}' \mu(t,\mathbf{x})U(\mathbf{x}-\mathbf{x}')\mu(t,\mathbf{x}') \right\}, \tag{12}$$

where $\mu(t,\mathbf{x})$ is an auxiliary field with respect to which $S\{f,\mu\}$ also has to be extremized. Variation of Eq. (12) with respect to $\mu(t,\mathbf{x})$ reproduces the KPZ equation (7) with

$$V(t,\mathbf{x}) = \int d\mathbf{x}' U(\mathbf{x}-\mathbf{x}')\mu(t,\mathbf{x}'), \tag{13}$$

whereas its variation with respect to $f(t,\mathbf{x})$ leads to

$$\partial\mu/\partial t + \mathrm{div}(\mathbf{u}\mu) + \nu\nabla^2\mu = 0, \tag{14}$$

where $\mathbf{u}(t,\mathbf{x}) \equiv \nabla f(t,\mathbf{x})/J$ is the "velocity" entering the Burgers equation (8). The form of Eq. (14) implies that the integral of $\mu(t,\mathbf{x})$ over whole space is a conserved quantity, whereas substitution of Eq. (13) into Eq. (11) shows that in terms of $\mu(t,\mathbf{x})$ the action can be rewritten as

$$S\{\mu\} = \frac{1}{2}\int_0^L dt \int\int d\mathbf{x}d\mathbf{x}' \mu(t,\mathbf{x})U(\mathbf{x}-\mathbf{x}')\mu(t,\mathbf{x}'). \tag{15}$$

In a system with δ-functional correlations, $U(x)=U_0\delta(\mathbf{x})$, V and μ differ from each other only by a constant factor U_0, and accordingly, Eq. (14) can be replaced by

$$\partial V/\partial t + \mathrm{div}(\mathbf{u}V) + \nu\nabla^2 V = 0. \tag{16}$$

If the beginning of a polymer (at $t=0$) is not fastened to a particular point and is free to fluctuate, the initial condition for the partition function $z(t,\mathbf{x})$ has to be chosen in the form $z(0,\mathbf{x})=\mathrm{const}$. In such a case to find the tails of $P_L(F)$, one has to find the solution of Eqs. (7) and (14) which satisfies the initial condition

$$f(0,\mathbf{x}) = 0, \tag{17}$$

and the final condition

$$f(L,0) = F, \tag{18}$$

where for the left tail $F<0$ and for the right tail $F>0$. Alternatively, condition (18) can be imposed by the inclusion of the δ-functional factor,

$$\int d\lambda \, \exp\big(i\lambda[f(L,\mathbf{x}=0) - F]\big), \tag{19}$$

into the functional integral defining the probability of a fluctuation. In such a case condition (18) for $f(L,\mathbf{x})$ should be replaced by the condition for $\mu(L,\mathbf{x})$,

$$\mu(L,\mathbf{x}) = \mu_0\delta(\mathbf{x}), \tag{20}$$

where, however, the value of $\mu_0 \propto \lambda$ has to be chosen to satisfy Eq. (18).

IV. FAR-LEFT TAIL

It turns out that in the case of the left tail the solution of Eqs. (7) and (14) which satisfies boundary conditions (17) and (18) can be constructed on the basis of the solution of these equations in which the potential V and all derivatives of f do not depend on t, which means that the time dependence of $f(t,\mathbf{x})$ is decoupled from its spacial dependence and is as trivial as possible,

$$f(t,\mathbf{x}) = E(t-t_1) + f(\mathbf{x}), \tag{21}$$

where $t_1=\mathrm{const}$ and $E=\mathrm{const}<0$. Below we for brevity call such solutions stationary.

For $f(t,\mathbf{x})$ of form (21) the replacement

$$f(\mathbf{x}) = -T \ln \Psi(\mathbf{x}) \tag{22}$$

transforms the KPZ equation (7) into a stationary Schrödinger equation,

$$E\Psi = \hat{H}\Psi, \tag{23}$$

for a single-particle quantum-mechanical problem defined by the Hamiltonian

$$\hat{H} = -\frac{T^2}{2J}\nabla^2 + V(\mathbf{x}), \tag{24}$$

where J plays the role of mass and T of Plank's constant $\hbar$ [compare with Eq. (5)]. On the other hand, when both $\mathbf{u}=-(T/J)\nabla\Psi/\Psi$ and μ do not depend on t, Eq. (14) is automatically fulfilled as soon as

$$\mu(\mathbf{x}) \propto \Psi^2(\mathbf{x}), \tag{25}$$

which implies

$$V(\mathbf{x}) = -\Lambda \int d\mathbf{x}' U(\mathbf{x}-\mathbf{x}')\Psi^2(\mathbf{x}'), \tag{26}$$

where Λ is an arbitrary constant. Substitution of Eq. (26) into Eq. (23) allows one to replace them by a single nonlinear Schrödinger equation,

$$E\Psi = -\frac{T^2}{2J}\nabla^2\Psi - \Lambda\Psi(\mathbf{x})\int_{-\infty}^{+\infty} d\mathbf{x}' U(\mathbf{x}-\mathbf{x}')\Psi^2(\mathbf{x}').$$

Equation (26) has been derived by Halperin and Lax[14] when looking for the optimal fluctuation of the potential $V(\mathbf{x})$, which for the given value of the ground-state energy $E<0$ of the quantum-mechanical Hamiltonian (24) minimizes the functional

$$s\{V\} = \frac{1}{2}\int\int d\mathbf{x}d\mathbf{x}' V(\mathbf{x})U^{-1}(\mathbf{x}-\mathbf{x}')V(\mathbf{x}'), \tag{27}$$

determining the probability of $V(\mathbf{x})$ (or, equivalently, minimizes E for the given value of $s\{V\}$). Apparently, in terms of our problem $s\{V\}$ is related to the action $S\{V\}$ defined by Eq. (11) as $S=Ls$. In the case of δ-functional correlations, $U(\mathbf{x})=U_0\delta(\mathbf{x})$, and t-independent potential $V(\mathbf{x})$, functional (11) is reduced to

$$S\{V\} = \frac{L}{2U_0}\int d\mathbf{x}V^2(\mathbf{x}). \tag{28}$$

A. δ-functional correlations, $d=1$

In a 1+1-dimensional system with a δ-correlated random potential, $U(x)=U_0\delta(x)$, the localized solution of Eqs. (23) and (26) (the soliton) exists for any $E<0$ and can be found exactly,[14]

$$\Psi(x) = \left(\frac{-2E}{\Lambda U_0}\right)^{1/2}\frac{1}{\cosh(x/\Delta)}, \tag{29}$$

$$V(x) = \frac{2E}{\cosh^2(x/\Delta)}, \tag{30}$$

where the length scale

$$\Delta = \frac{T}{(-2JE)^{1/2}} \tag{31}$$

can be called soliton width. This allows one to conclude that the stationary solution of Eqs. (7) and (16) is given by Eq. (30) and

$$f(t,x) = E(t-t_1) + T\ln\left(2\cosh\frac{x}{\Delta}\right), \tag{32}$$

which follows from the substitution of Eq. (29) into Eqs. (21) and (22). Note that in Eq. (32) the constant t_1 has been redefined in order to absorb Λ.

Differentiation of Eq. (32) with respect to x gives a stationary profile of $u(x)$,

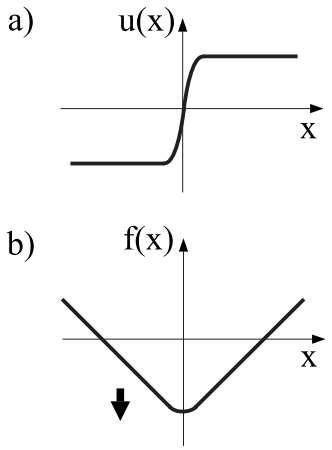

FIG. 1. The spatial dependence of u and f in the stationary solution of Eqs. (7) and (16).

$$u(x) = v\tanh\frac{x}{\Delta}, \tag{33}$$

schematically shown in Fig. 1(a). Here

$$v = T/J\Delta \tag{34}$$

is the velocity of the outward flow created by the forces acting inside the soliton. The profile (33) up to a sign coincides with the one in a stationary shock wave with the same amplitude v. The solitons of such a kind (both stationary and moving) have been discussed in a number of works by Fogedby.[15]

The stationary profile of f described by Eq. (32) is schematically shown in Fig. 1(b). With the increase in time it is moving downward as a whole with a constant velocity $\partial f/\partial t=E$. Away from the soliton's core, that is, at $|x|\gg\Delta$, the dependence described by Eq. (32) can be approximated as

$$f(t,x) \approx E(t-t_1) + (-2JE)^{1/2}|x|. \tag{35}$$

Since Eq. (35) describes a solution of the noiseless KPZ equation, its form does not depend on the form (or amplitude) of the random potential correlator $U(x)$.

The stationary solution minimizes the action for the given negative value of $\partial f/\partial t=E$. Therefore, it allows one to find the optimal value of S in situations when it is not influenced by the initial condition. Substitution of Eq. (30) into Eq. (28) then gives

$$S(\Delta) = \frac{2}{3}\frac{T^4L}{U_0J^2\Delta^3}. \tag{36}$$

Apparently, the condition $f(L,0)-f(0,0)=F$ is fulfilled when $E=F/L$, which corresponds to

$$F = -\frac{T^2}{2J\Delta^2}L \tag{37}$$

and

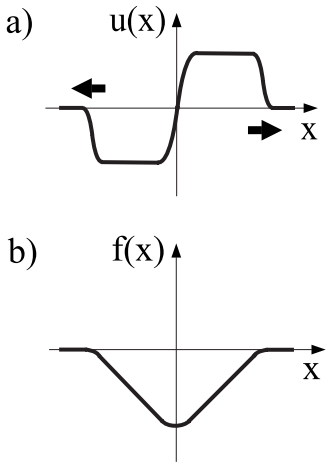

FIG. 2. The spatial dependence of u and f in the solution of Eqs. (7) and (16) corresponding to the left tail of $P_L(F)$. The arrows show the directions of motion of the two shock waves.

$$S(F) = \frac{4\sqrt{2}}{3}\frac{T(-F)^{3/2}}{U_0 J^{1/2} L^{1/2}}. \tag{38}$$

However, the real optimal fluctuation also has to respect the initial condition and it is clear that the spacial dependence of f in Eq. (32) in no way resembles the initial condition (17). In terms of the quantum-mechanical problem with time-independent potential $V(\mathbf{x})$, it is clear that the applicability of the relation $F \approx EL$ requires to have $(\delta E)L \gg T$, where δE is the energy gap separating the ground state of the Hamiltonian (24) from the first excited state. Since in potential (30) there exists only one bound level with a negative energy,[16] whereas excited states can have any non-negative energy, this condition is equivalent to $-F \gg T$.

For constructing a nonstationary solution which eliminates the inconsistency between the forms of the stationary solution and of the initial condition (without increasing the action), one has to complement the soliton shown in Fig. 1(a) by two traveling shock waves [as shown in Fig. 2(a)], whose existence does not require any additional pumping. Both these shock waves will be moving outwards with velocity $v/2$. Their presence will change the profile of $f(t,x)$ to the one shown in Fig. 2(b) so that $f(x)$ will be given by Eq. (32) only in the interval where $f(t,x) < 0$, whereas outside of this region it will coincide with the initial condition (17) (with a smooth crossover between the two solutions). This means that if a potential localized in the vicinity of $x=0$ is switched on at $t=0$, its influence on $f(t,x)$ at $t>0$ extends only to a finite (but growing with t) region, which is perfectly logical.

In such a situation the constant t_1 in Eq. (32) [or in Eq. (35)] will have the meaning of an effective time required for the formation of the nonstationary solution shown in Fig. 2. At the initial stage, that is, at $t \lesssim t_1$, the spacial distribution of $V(x)$ will substantially differ from the one given by Eq. (30) The value of t_1 can be estimated from the comparison of the soliton width $\Delta = 2\nu/v$ with the velocity v of the flow it creates, which gives $t_1 \sim \nu/v^2$. This allows one to expect that for $L \gg t_1$ the main contribution to the action is coming from the region in t where Eq. (30) gives a sufficiently accurate description of the solution, and therefore the value of $S(F)$ is given by Eq. (38). In terms of F the constraint $L \gg t_1$ corresponds to the condition $-F \gg T$ (which was already derived above in different terms).

The same condition allows us to neglect the final stage of the optimal-fluctuation evolution. At this stage the potential has to shrink from the form given by Eq. (30) to a δ-function as suggested by Eq. (20). Simultaneously, the downward tip of $f(x)$ has to change its shape from rounded to more sharp. The decrease in $f(x=0)$ related to this process can be expected to be comparable with the change in f induced by the rounding of the tip, which according to Eq. (32) is of the order of T and therefore for $-F \gg T$ can be ignored.

Note that the same answer for $S(F)$, Eq. (38), can be also obtained in the framework of the Kardar-Zhang replica approach based on mapping a system to a set of interacting bosons and keeping only the ground-state contribution to the partition function of these bosons (see Appendix A for more details). In Appendix B we demonstrate that the change in the initial condition from free to fixed does not change the form of the main contribution to $S(F)$. The same conclusion is even more easily attained in terms of the replica approach (see Appendix A).

In the remaining part of this section, we analyze the systems with an arbitrary dimension and/or finite-range correlations assuming that the main features of the optimal fluctuation determining the far-left tail are the same. Namely, we expect that in a growing region around $\mathbf{x}=0$, the solution is close to the stationary solution, whereas outside of this region it is close to the initial condition $f(t,\mathbf{x})=0$, the crossover between the two regions being described by a corresponding solution of the noiseless KPZ equation. In such a situation the action of the optimal fluctuation is determined by the form of the stationary solution.

B. Generalization to $d \neq 1$

If the dimension of the transverse space d is not equal to 1, the joint solution of Eqs. (23) and (26), that is, the wave-function $\Psi(\mathbf{x})$ which minimizes the sum of a positive kinetic energy,

$$\mathcal{K} \equiv \frac{T^2}{2J}\frac{\int d^d\mathbf{x}|\nabla\Psi(\mathbf{x})|^2}{\int d^d\mathbf{x}|\Psi(\mathbf{x})|^2}, \tag{39}$$

and a negative potential energy,

$$\mathcal{V} \equiv \frac{\int d^d\mathbf{x}V(\mathbf{x})|\Psi(\mathbf{x})|^2}{\int d^d\mathbf{x}|\Psi(\mathbf{x})|^2}, \tag{40}$$

for a given value of the functional $S\{V\}$ defined by Eq. (28), cannot be found exactly. However, in a situation when this wave function and, therefore, the potential $V(\mathbf{x}) \propto -U_0\Psi^2(\mathbf{x})$ are well localized at some length scale Δ, an estimate for Δ and a qualitative relation between S and F can be obtained without finding the exact form of $\Psi(\mathbf{x})$.

When $\Psi(\mathbf{x})$ can be characterized by a single relevant length scale Δ, one has

$$\mathcal{K}(\Delta) \sim \frac{T^2}{J\Delta^2}, \tag{41}$$

whereas the absolute value of $\mathcal{V} \sim V(0)$ at a given S can be estimated with the help of Eq. (28), which gives

$$S \sim \frac{L}{U_0}\Delta^d \mathcal{V}^2, \tag{42}$$

and therefore,

$$\mathcal{V}(\Delta) \sim -\left(\frac{SU_0}{L\Delta^d}\right)^{1/2}. \tag{43}$$

For $0<d<4$ the sum $\mathcal{K}(\Delta)+\mathcal{V}(\Delta)$ has a minimum with respect to Δ when $\mathcal{K}(\Delta) \sim -\mathcal{V}(\Delta)$ and therefore both $\mathcal{K}$ and $-\mathcal{V}$ have to be of the same order as $-E=-F/L$.

Substitution of $\mathcal{V} \sim F/L$ into Eq. (42) allows one to rewrite this relation as

$$S \sim \frac{\Delta^d F^2}{U_0 L}. \tag{44}$$

On the other hand, an estimate for Δ in terms of F can be obtained from the relation $\mathcal{K} \sim -E$, which gives

$$\Delta(F) \sim \frac{T}{2J}\left[\frac{JL}{-F}\right]^{1/2}. \tag{45}$$

After that to obtain an estimate for $S(F)$ one needs only to substitute Eq. (45) into Eq. (44), which leads to

$$S(F) \sim \frac{T^d(-F)^{2-d/2}}{U_0 J^{d/2} L^{1-d/2}}. \tag{46}$$

Naturally, for $d=1$ Eq. (46) is consistent with Eq. (38) derived in Sec. IV A on the basis of the exact solution of Eqs. (23) and (26).

For $d>4$ the sum of $\mathcal{K}(\Delta)$ and $\mathcal{V}(\Delta)$ at a given S is not bounded from below and tends to $-\infty$ when $\Delta \to 0$. Accordingly, for any $F<0$ it becomes possible to find a stationary fluctuation with an arbitrary low action, so the method of optimal fluctuation is no longer applicable. However, it turns out that the range of the applicability of Eq. (46) is even more narrow than the interval $0<d<4$, where the action of stationary fluctuations has a well-defined positive minimum.

The point is that L enters Eq. (46) as the total time of the development of the optimal fluctuation of $f(t,\mathbf{x})$. From this it is clear that Eq. (46) can be expected to be valid only if S decreases with the increase in L, which forces the time of the development of the optimal fluctuation to coincide with L. In the opposite case (when S decreases with the *decrease* in L) there appears a possibility to decrease the action of the fluctuation we are considering by making the time of its development smaller than L. Namely, if one makes in Eq. (44) a replacement

$$L \Rightarrow \gamma^2 L, \quad \Delta \Rightarrow \gamma\Delta \tag{47}$$

conserving relation (45), this leads to $S \Rightarrow \gamma^{d-2}S$. Therefore, for $d>2$ a consistent decrease in the size of the fluctuation and in the time of its development allow one to make $S(F)$ arbitrarily small by choosing a sufficiently small γ. This suggests that the result (46) can be expected to be applicable only at $0<d<2$, whereas at $d>2$ the optimal fluctuation corresponding to the most distant part of the left tail has to be localized at small scales and its form has to be determined by the form of a cutoff. Without a cutoff the problem with $d \geq 2$ and δ-functional correlations is ill-defined.

Note that at $d \geq 2$, the problem with δ-functional correlations is ill-defined also for another reason. Namely, at $d \geq 2$ the perturbative corrections to the viscosity ν and other quantities acquire ultraviolet divergencies which at $d<2$ are absent. Apparently, this is not a coincidence but another manifestation of the same phenomenon. Therefore, for $d \geq 2$ some ultraviolet cutoff must be introduced into the problem. One of the most natural ways to do it consists in assuming that the correlations of a random potential are characterized by a finite correlation radius.

C. Finite-range correlations

When random potential correlator $U(\mathbf{x})$ [which we assume to be spherically symmetric, $U(\mathbf{x}) \equiv U(|\mathbf{x}|)$] is characterized by a finite correlation radius ξ, the stationary solution of Eqs. (7) and (14) cannot be found exactly even at $d=1$. However, it is clear from the form of Eq. (13) relating V and μ that when the soliton width Δ is much larger than ξ, the actual solution has to be rather close to the solution for $\xi=0$, the same being true also for the value of $S(F)$. It follows from Eq. (45) that in terms of F the condition $\Delta \gg \xi$ corresponds to

$$-F \ll F_\xi \sim \frac{T^2 L}{J\xi^2}. \tag{48}$$

It turns out that for the opposite relation between the parameters, $-F \gg F_\xi$, the stationary solution of Eqs. (7) and (14) also can be found rather accurately. As it is shown below, in such a case μ is localized in a region which is much narrower than ξ, whereas both f and V change at the scales of the order of ξ. In particular, it follows from Eq. (13) that in such a situation the spacial dependence of the potential $V(\mathbf{x})$ just repeats that of $U(\mathbf{x})$,

$$V(\mathbf{x}) \approx -U(\mathbf{x})\varepsilon, \tag{49}$$

whereas the amplitude of $V(\mathbf{x})$ is determined by

$$\varepsilon \equiv -\int d\mathbf{x}\mu(\mathbf{x}), \tag{50}$$

the overall strength of the negative potential source $\mu(\mathbf{x})$.

For $-F \gg F_\xi$ the viscous term in Eq. (7) can be neglected, which immediately gives that in the spherically symmetric stationary solution,

$$\partial f/\partial t \approx -U(0)\varepsilon \tag{51}$$

and

$$\left(\frac{\partial f}{\partial r}\right)^2 = 2J[U(0)-U(r)]\varepsilon, \tag{52}$$

where $r=|\mathbf{x}|$ so that

$$f(\mathbf{x}) = f(0) + \sqrt{2J\varepsilon}\int_0^{|\mathbf{x}|} dr\sqrt{U(0) - U(r)}. \tag{53}$$

In terms of the Schrödinger equation (23) the neglect of the viscous term in the stationary KPZ equation corresponds to nothing else but using the semiclassical approximation for the calculation of the ground-state wave function.

Substitution of $\Psi(\mathbf{x}) = \exp[-f(\mathbf{x})/T]$ with $f(\mathbf{x})$ given by Eq. (53) into Eq. (25) demonstrates that at $|\mathbf{x}| \ll \xi$,

$$\mu(\mathbf{x}) \propto \exp\left[-\frac{\mathbf{x}^2}{2\Delta^2}\right], \tag{54}$$

where Δ, the width of the region where the potential source $\mu(\mathbf{x})$ is localized, is given by

$$\Delta(F) = \left[\frac{T^2U(0)}{-4U_{rr}(0)JE}\right]^{1/4} \sim \left(\frac{F_\xi}{-F}\right)^{1/4}\xi. \tag{55}$$

When deriving this estimate we have replaced $-U_{rr}(0)$ by $U(0)/\xi^2$ and E by F/L. The result shows that the assumption $\Delta(F) \ll \xi$, which has been used above to obtain Eq. (49), is indeed self-consistent as soon as $-F \gg F_\xi$.

Substitution of Eq. (49) into Eq. (15) reduces the expression for the action to a very simple form,

$$S = \frac{U(0)}{2}L\varepsilon^2 = \frac{LE^2}{2U(0)}, \tag{56}$$

which is easily recognizable to those familiar with application of the optimal-fluctuation approach to a quantum-mechanical problem with finite-range correlations of a random potential[17] and after substitution of $E = F/L$ gives

$$S(F) = \frac{F^2}{2U(0)L}. \tag{57}$$

The same temperature-independent answer can be also reproduced in terms of the Kardar-Zhang replica approach (see Appendix A).

Thus we have demonstrated that for $\xi > 0$ the most distant part of the left tail is Gaussian independently of the dimension. Since the width of the region where μ is localized grows with the decrease in $-F$, a crossover to some other regime must occur when this width becomes comparable with ξ. In particular, for $d < 2$ and $\xi \ll x_0$ the dependence of S on F at $-F \ll F_\xi$ has to be described by Eq. (46) with a subsequent crossover to the universal regime discussed in Sec. VI B. Naturally, the increase in ξ (or d) leads to shrinking and subsequent vanishing of the region where $S(F)$ can be described by Eq. (46). On the other hand, when ξ is taken to zero F_ξ goes to infinity, which leads to the disappearance of the region with Gaussian behavior.

D. Boundary from below

It is worthwhile to emphasize that expression (57) gives an exact boundary from below for the value of $S(F)$ in the optimal fluctuation. This is so because the potential of the form

$$V(\mathbf{x}) = \frac{U(\mathbf{x})}{U(0)}V(\mathbf{x} = 0) \tag{58}$$

minimizes functional (27) for the given value of $V(\mathbf{x}=0)$, from where

$$S(F) \geq \frac{1}{2U(0)}\int_0^L dt[V(t,0)]^2. \tag{59}$$

On the other hand, in a growing fluctuation of $f(t,\mathbf{x})$ which has a spherically symmetric shape and an extremum at $\mathbf{x}=0$, the absolute value of $\partial f(t,0)/\partial t$ is bounded from above by $|V(t,0)|$ because at the point of extremum the second term in the left-hand side of the KPZ equation (7) vanishes, whereas the third term, $-\nu\nabla^2 f$, has to have the same sign as $\partial f(t,0)/\partial t$. This allows one to conclude that

$$S(F) \geq \frac{1}{2U(0)}\int_0^L dt\left[\frac{\partial f(t,0)}{\partial t}\right]^2 \geq \frac{F^2}{2U(0)L}. \tag{60}$$

Apparently, this inequality is reduced to equality only if (i) $V(t,\mathbf{x})$ is of form (58), (ii) the viscous term in the KPZ equation can be neglected, and (iii) $\partial f(t,0)/\partial t$ does not depend on time. Since in the negative fluctuation of f considered in Sec. IV C all these conditions are satisfied rather accurately, the action of this fluctuation is approximately equal to the boundary from below given by Eq. (60).

Note that the argument leading to the derivation of Eq. (60) is valid for both signs of F. Therefore inequality (60) has to be satisfied also in the far-right tail.

V. FAR-RIGHT TAIL

Our analysis has established that the optimal fluctuation corresponding to the left tail of $P_L(F)$ has a very special shape which can be characterized by two different scales. Namely, the size of the area where the potential V is localized, $\Delta(F)$, is much smaller then the total size of the fluctuation $\tilde{\Delta}(F) \sim (-FL/J)^{1/2}$, that is, the width of the area where f and $\mathbf{u}$ essentially deviate from zero. Apparently this property is closely related to the fact that inside a growing negative fluctuation of f the terms $\partial f/\partial t$ and $(1/2J)(\nabla f)^2$ in the functional,

$$S\{f\} = \frac{1}{2U_0}\int_0^L dt\int d\mathbf{x}\left[\frac{\partial f}{\partial t} + \frac{1}{2J}(\nabla f)^2 - \nu\nabla^2 f\right]^2, \tag{61}$$

defining the probability of a fluctuation in a system with a δ-correlated potential have to be of the opposite signs. This provides a possibility for their mutual compensation in almost the whole volume of the fluctuation. It is clear that in the case of the right tail such a cancellation is impossible because in the substantial part of the optimal fluctuation $\partial f/\partial t$ has to be of the same sign as $(1/2J)(\nabla f)^2$. As a consequence, the optimal fluctuation corresponding to the right tail must have a shape which can be characterized by a single relevant length scale, $\Delta_+(F)$.

This length scale can be estimated from the comparison of $\partial f/\partial t \sim F/L$ with $(1/2J)(\nabla f)^2 \sim F^2/J\Delta_+^2$, which shows that Δ_+ has to be of the same order as the total size of the optimal fluctuation with $F < 0$,

$$\Delta_+(F) \sim \tilde{\Delta}(-F) \sim \left(\frac{LF}{J}\right)^{1/2}. \tag{62}$$

Note that for $\Delta_+(F)$ given by Eq. (62) the viscous term in the integrand of functional (61) can be neglected if F is large enough. This is precisely the reason why an estimate for Δ_+ can be obtained by matching the two other terms in this integrand. A comparison of $\nu\nabla^2 f \sim \nu F/\Delta_+^2$ with $(1/2J)(\nabla f)^2 \sim F^2/J\Delta_+^2$ shows that the condition which allows one to neglect the viscous term can be written as $F \gg 2J\nu = T$. Apparently this constraint is automatically fulfilled as soon as one considers the most distant part of the tail.

Substitution of Eq. (62) into the relation

$$S \sim \frac{L\Delta_+^d}{U_0}\left(\frac{F}{L}\right)^2, \tag{63}$$

following from the assumption that $\Delta_+(F)$ is the only relevant length scale in the problem gives then an estimate for the action determining the form of the far-right tail of $P_L(F)$,

$$S(F) \sim \frac{F^{2+d/2}}{U_0 J^{d/2} L^{1-d/2}}, \tag{64}$$

which naturally is independent of temperature. On a more formal level, the same relation can be obtained as a variational estimate from above. If one assumes, for example, that

$$f(t,\mathbf{x}) = \frac{Ft}{L}\exp\left(-\frac{\mathbf{x}^2}{2\Delta_+^2}\right) \tag{65}$$

and substitutes Eq. (65) into Eq. (61), then for $0<d<4$ the result of this substitution $S_{\mathrm{var}}(\Delta_+)$ (which for $F \gg T$ is insensitive to the presence of the viscous term in the integrand) has a minimum with respect to the variational parameter Δ_+. This minimum is situated at $\Delta_+(F)$ satisfying relation (62), whereas the value of $S_{\mathrm{var}}[\Delta_+(F)]$ satisfies relation (64).

The important difference between the far-left and far-right tails is that in the far-right tail, the width of the region where the fluctuation of a random potential is localized grows with the increase in $|F|$. In such a situation one can expect that the shape of the optimal fluctuation in the most distant part of the tail at finite ξ will be the same as for a δ-correlated potential. This requires the fulfillment of the condition $\Delta_+ \gg \xi$, that is, $F \gg J\xi^2/L$. Therefore, for a given ξ and sufficiently large L the region of the applicability of Eq. (64) will be extended to the whole nonuniversal part of the right tail.

Although the minimum of $S_{\mathrm{var}}(\Delta_+)$ with respect to Δ_+ exists for any d in the interval $0<d<4$, it follows from the form of Eq. (64) that this equation can be expected to be directly applicable only at $d<2$, exactly like in the case of the analogous expression for the far-left tail, Eq. (46). For $d>2$ Eq. (64) (where L enters as the total time of the development of the fluctuation) predicts that the action can be decreased by making the time of the development of this fluctuation much smaller than L. According to Eq. (62) this will be accompanied by the decrease in the size of the fluctuation. This suggests that at $d>2$ the optimal fluctuation must have a different structure, which has to be sensitive to the form of a random potential correlator at small lengths.

If the first factor in the right-hand side of Eq. (65) is replaced by

$$\frac{F\sinh(t/L_+)}{\sinh(L/L_+)},$$

which allows one to vary not only the characteristic size of a fluctuation Δ_+ but also the time of its development L_+, then for $U(\mathbf{x}) \propto \exp(-\mathbf{x}^2/2\xi^2)$ and $d>2$ the minimum of the action is achieved at $\Delta_+ \sim \xi$ and $L_+ \sim J\xi^2/F$, which corresponds to

$$S(F) \sim \frac{\xi^{d-2}F^3}{U_0 J}. \tag{66}$$

Note that at $d=2$, the estimates given by Eqs. (64) and (66) coincide with each other. Naturally, at the marginal dimension of $d=2$ (where algebraic divergences are replaced by logarithmic) some logarithmic factors may appear in the expression for the action.

VI. MODIFICATION OF TAILS BY THE RENORMALIZATION EFFECTS

In terms of the Burgers equation parameters (the viscosity $\nu = T/2J$ and the pumping force intensity $D = U_0/2J^2$), Eq. (46) can be rewritten as

$$S(F) \sim \frac{\nu^d}{D}\frac{(-F/J)^{2-d/2}}{L^{1-d/2}}. \tag{67}$$

This estimate has been derived at $d<2$ and $\xi=0$ and is applicable also at $\xi>0$ as soon as $\xi \ll \Delta$. However, from the nature of the optimal-fluctuation approach, it is clear that the range of the applicability of Eq. (67) is restricted also from the other side because in order to disregard the renormalization of any parameters by the nonlinearity the soliton has to be sufficiently narrow: $\Delta \ll x_0$, where x_0 is defined by the relation

$$x_0^{2-d} \sim \frac{\nu^3}{D} \sim \frac{T^3}{JU_0}. \tag{68}$$

At any $d \neq 2$ x_0 is the only parameter with the dimension of $\mathbf{x}$ which can be constructed from T, J, and U_0. In particular, in the case of $d<2$ and small ξ we are discussing now, x_0 is the length scale at which the perturbative corrections to ν and D become comparable with the bare values of these parameters.

Thus, at $\Delta \gg x_0$ the renormalization effects become important. In such a regime the probability of a large negative fluctuation of F is determined not by a single fluctuation (and small deviations from it) but by a relatively wide class of fluctuations, the summation over which can be taken into account by analyzing an optimal fluctuation in a system with renormalized parameters. Since in all the cases we consider the optimal fluctuations are quasistationary (see below) and well localized at a particular length scale, this can be done by replacing all parameters in Eq. (67) by their effective values at the corresponding length scale and zero frequency.[8] However, it is well known that only ν and D are subject to renor-

malization, whereas the amplitude of the nonlinear term in the KPZ equation (7) (and, therefore, the coefficient J) cannot be renormalized as a consequence of the Galilean invariance.[18]

From the continuity it is clear that when the instanton is not too narrow, the approach relying on using Eq. (67) with renormalized parameters can be also expected to work even at $d \geq 2$ [where Eq. (67) has no region of the direct applicability] as soon as the parameters of the system correspond to the same phase as at $d<2$ [namely, the strong-coupling phase in which the fluctuations of $f(t,\mathbf{x})$ in a stationary situation are divergent, see Eq. (69) below]. At $d>2$ this requires to have $x_0/\xi > \kappa(d)$, that is, the temperature T should be lower then some critical value $T_c(d)$,[19] which tends to infinity when $d \to 2+0$. In the weak-coupling phase, that is, at $T>T_c(d)$, typical fluctuations of $f(t,\mathbf{x})$ in the stationary situation can be described by neglecting the nonlinear term in the KPZ equation (7). However, the form of the most distant parts of the tails of $P_L(F)$ is insensitive to the relation between T and $T_c(d)$ and in both phases has to be given by Eqs. (57) and (66). To describe how the renormalization effects change the shape of the tails of $P_L(F)$ in the regime when they are important (which corresponds to the universal parts of the tails in the strong-coupling phase), we first have to review some known properties of the stationary solution of the KPZ model in the strong-coupling regime.

A. Stationary solution of the KPZ model

In a stationary situation the divergence of fluctuations in the strong-coupling phase of a KPZ system is algebraic. Their behavior at large scales in space-time can be described by two fundamental exponents,[18,20]

$$\langle [f(t,\mathbf{x}) - f(t,\mathbf{x}')]^2 \rangle \propto |\mathbf{x}-\mathbf{x}'|^{2\chi} g\left(\frac{|t-t'|}{|\mathbf{x}-\mathbf{x}'|^z} \right). \tag{69}$$

Here $\chi \equiv \chi(d)$ is the roughening exponent characterizing the equal-time interface fluctuations, $z \equiv z(d)$ is the dynamic exponent describing the scaling of the relaxation time with the length-scale, whereas the function $g(\alpha)$ has a finite limit at $\alpha \to 0$ and diverges as $\alpha^{2\chi/z}$ when $\alpha \to \infty$. It is well known[18] that the existence of the Galilean invariance imposes

$$z + \chi = 2. \tag{70}$$

At $d=1$ the value of the exponent $\chi = 1/2$ is known exactly because the equal-time correlator of $f(t,\mathbf{x})$ in a system with δ-functional correlations of a random potential has to be exactly the same as in the absence of the nonlinearity.[12] This property is a consequence of the fluctuation-dissipation theorem,[21] which is obeyed by Eq. (7) only at $d=1$. At $d \neq 1$ the values of the exponents z and χ are known only from approximate or numerical calculations. In terms of the directed polymer problem the dependence (69) corresponds to

$$\langle [\mathbf{x}(t) - \mathbf{x}(t')]^2 \rangle \propto (t-t')^{2/z}, \tag{71}$$

which shows that $\zeta = 1/z$ plays the role of the roughening exponent for the transverse displacements inside an infinite polymer and therefore cannot be smaller than 1/2,[22] from where $z \leq 2$.

A natural way to describe the effective renormalization of ν and D by the nonlinearity consist in introducing[23] a generalized viscosity $\nu(\omega,\mathbf{q})$ and a generalized pumping intensity $D(\omega,\mathbf{q})$ defined by the relations

$$G(\omega,\mathbf{q}) = [-i\omega + \nu(\omega,\mathbf{q})q^2]^{-1}, \tag{72}$$

$$C(\omega,\mathbf{q}) = 2J^2 |G(\omega,\mathbf{q})|^2 D(\omega,\mathbf{q}), \tag{73}$$

where $G(\omega,\mathbf{q})$ and $C(\omega,\mathbf{q})$ are, respectively, the Fourier transforms of the response function and of the two-point correlation function of $f(t,\mathbf{x})$. The form of Eqs. (72) and (73) corresponds to the replacement of the considered nonlinear system by a linear system with the same form of $G(\omega,\mathbf{q})$ and $C(\omega,\mathbf{q})$.

The compatibility with the behavior described by Eq. (69) requires then that at small enough q,

$$\lim_{\omega \to 0} \nu(\omega,\mathbf{q}) \propto q^{-(2-z)}, \quad \lim_{\omega \to 0} D(\omega,\mathbf{q}) \propto q^{-(d+2\chi-z)}.$$

This suggests that the behavior of low-frequency fluctuations with typical or smaller amplitude can be qualitatively described by using an effective viscosity $\nu_{\mathrm{eff}}(R)$ and an effective pumping intensity $D_{\mathrm{eff}}(R)$ which algebraically depend on a length scale R,

$$\nu_{\mathrm{eff}}(R) \sim \nu \left(\frac{R}{a_\nu} \right)^{2-z}, \quad D_{\mathrm{eff}}(R) \sim D \left(\frac{R}{a_D} \right)^{4+d-3z}, \tag{74}$$

where in accordance with Eq. (70) we have replaced χ by $2-z$. As a convenient way of describing the amplitudes of $\nu_{\mathrm{eff}}(R)$ and $D_{\mathrm{eff}}(R)$, we have introduced in Eqs. (74) two new length scales, a_ν and a_D. For $d=1$ and $\xi \lesssim x_0$ both a_ν and a_D can be expected to be of the order of x_0 because in such a situation x_0 is the only relevant length in the problem. However for $\xi \gg x_0$ and/or $d>1$ these two length scales do not have to be of the same order. Since both ν and D increase under the renormalization, Eqs. (74) can be expected to be applicable only for $R \gg a_\nu, a_D$.

In scaling regime, when $\nu_{\mathrm{eff}}(R)$ and $D_{\mathrm{eff}}(R)$ behave themselves in accordance with Eqs. (74), both these quantities have no direct relation to their bare values, ν and D. Their origin can be traced to the effect of fluctuations with shorter wavelengths than the given length scale R. In particular, it follows from the structure or the KPZ equation (7) that at the length scale R the role of the effective random potential is played by the deviation of $-(J/2)\langle \mathbf{u}^2 \rangle_R$ from its average value, $-(J/2)\overline{\langle \mathbf{u}^2 \rangle_R}$, where $\langle \cdots \rangle_R$ denotes spatial averaging over a region with a linear size of the order of R. From this the value of $D_{\mathrm{eff}}(R)$ can be estimated as

$$D_{\mathrm{eff}}(R) \sim \int_{|\mathbf{r}|<R} d\mathbf{r} \int_{-\infty}^{+\infty} d\tau \overline{[u^a(t,\mathbf{x}) u^b(t+\tau, \mathbf{x}+\mathbf{r})]^2} \sim \frac{R^{2+d} u_{\mathrm{typ}}^4(R)}{\nu_{\mathrm{eff}}(R)}. \tag{75}$$

In Eq. (75) we have assumed that the integration over $d\tau$ can be replaced by the multiplication by the factor $\sim \tau(r)$, where

$\tau(r) \sim r^2/\nu_{\rm eff}(r)$ is the characteristic relaxation time which can be associated with the length scale r, whereas the result of the integration over $d\mathbf{r}$ has been estimated assuming that as a consequence of the universality for any length scale there exist only one characteristic velocity scale which can be associated with this length scale (in other terms, there is no anomalous scaling). We have chosen as such a velocity scale the typical velocity, $u_{\rm typ}(R)$, defined by the relation

$$u_{\rm typ}^2(R) \equiv \overline{\langle \mathbf{u} \rangle_R^2} \sim \frac{D_{\rm eff}(R)}{\nu_{\rm eff}(R) R^d}. \tag{76}$$

Substitution of Eq. (76) into Eq. (75) then gives the relation

$$\frac{\nu_{\rm eff}^3(R)}{D_{\rm eff}(R)} \sim R^{2-d}, \tag{77}$$

whose structure is analogous to that of Eq. (68). The consistency between Eqs. (77) and (70) confirms the correctness of assumptions, which have been used for the derivation of Eq. (77). In terms of the length scales a_ν and a_D introduced above, see Eqs. (74), relation (77) can be rewritten as

$$a_\nu \sim \left(\frac{x_0}{a_D}\right)^{(2-d)/3(2-z)} a_D, \tag{78}$$

which for $d=1$ (when $z=3/2$) is reduced to

$$a_\nu \sim (x_0^2 a_D)^{1/3}. \tag{79}$$

When the dynamics of fluctuations at $R \sim \xi$ is dominated by wave breaking, the value of $u_{\rm typ}(\xi)$ can be estimated as a characteristic velocity $u_\xi \sim (D\tau_\xi/\xi^3)^{1/2}$, which is created by a random force with characteristic length scale ξ during the time $\tau_\xi \sim \xi/u_\xi$ required for breaking of such a fluctuation, which gives $u_\xi \sim (D/\xi^{1+d})^{1/3}$. A comparison of this estimate with Eqs. (75) and (76) suggests that in such a regime $D_{\rm eff}(\xi) \sim D$, that is, $a_D \sim \xi$. At $d<2$ we expect this conclusion to be applicable when $\xi \gtrsim x_0$, whereas at $d>2$ in the whole region of the existence of the strong-coupling phase.

It follows from the definition of $u_{\rm typ}(R)$ that with the increase in R the value of $u_{\rm typ}(R)$ has to decrease. A comparison of Eq. (76) with Eqs. (74) allows one then to conclude that z has to be larger than 1.

B. Universal part of the left tail

After replacing in Eq. (45) $T/2J \equiv \nu$ by $\nu_{\rm eff}(\Delta)$, one obtains a relation which allows one to find that in the regime when the renormalization effects are important the estimate for the soliton width Δ acquires a form

$$\Delta \equiv \Delta(F) \sim a_\nu \left(\frac{L\nu^2 J}{-Fa_\nu^2}\right)^{1/2(z-1)}. \tag{80}$$

A substantial change in Δ in comparison with what is given by Eq. (45) means that in the regime we consider now the probability of a large negative fluctuation of F is determined not by a narrow vicinity of the fluctuation which minimizes the original action (like it happens in the more distant part of a tail), but by a wide vicinity of an essentially different fluctuation whose dominance is ensured by a factor related to the integration over its vicinity. In the framework of a renormalization-group approach, this factor is effectively taken into account when one is replacing different parameters by their renormalized values.

An estimate for the action can be then obtained by making in Eq. (67) a replacement

$$\nu \to \nu_{\rm eff}(\Delta), \quad D \to D_{\rm eff}(\Delta), \tag{81}$$

with Δ given by relation (80). With the help of Eq. (77) the result of this substitution can be reduced to the form

$$S(F) \sim \left(\frac{-F}{F_*}\right)^\eta, \tag{82}$$

with exponent

$$\eta = \eta_- \equiv \frac{z}{2(z-1)}, \tag{83}$$

which depends on d only through the dynamic exponent $z \equiv z(d)$ but not explicitly. Here

$$F_* \sim J\nu \left(\frac{\nu L}{a_\nu^2}\right)^\omega \tag{84}$$

plays the role of a characteristic free-energy scale whose dependence on L is described by the exponent

$$\omega = 1 - \frac{1}{\eta_-} = \frac{2}{z} - 1. \tag{85}$$

The universality hypothesis for the directed polymer problem[24] (or, more generally, for the collective pinning problem[25]) suggests that F_* has to be of the same order as a characteristic elastic energy $E_{\rm el} \sim J(\delta x)^2/L$, where the dependence of the characteristic transversal displacement between the two ends of a polymer, $\delta x \equiv |\mathbf{x}(t=L) - \mathbf{x}(t=0)| \propto L^\zeta$, on its total length L is described by the roughening exponent ζ so that

$$\omega = 2\zeta - 1. \tag{86}$$

A comparison of Eq. (86) with Eq. (85) demonstrates that the fluctuations of δx are described by the same roughening exponent $\zeta = 1/z$ as fluctuations inside an infinite polymer, see Eq. (71), in full agreement with what one expects from the universality. This consistency can be considered as an additional confirmation of the validity of the set of assumptions which have been used for obtaining Eq. (82).

Note that the list of these assumptions includes the conjecture that the system evolves sufficiently slow so that at relevant length scales, it can be considered as already equilibrated, which is a necessary condition for using Eqs. (74). For this the total evolution time L has to be much larger then the characteristic relaxation time $\tau(\Delta) \sim \Delta^2/\nu_{\rm eff}(\Delta)$, which can be associated with the length scale Δ.[26] Since in terms of $\Delta(F)$ and L relation (82) can be rewritten as

$$S(F) \sim \frac{\nu_{\text{eff}}(\Delta)}{\Delta^2} L \sim \frac{L}{\tau(\Delta)}, \tag{87}$$

the constraint $L \gg \tau(\Delta)$ is equivalent to $S(F) \gg 1$ and, accordingly, is automatically fulfilled as soon as one is dealing with the tail.

It is also important that the effective viscosity and effective pumping intensity given by Eqs. (74) and following from the form of the correlation function (69) can be used for the description only of typical (or more weak) fluctuations. The comparison of the characteristic velocity of the flow created around the instanton,

$$u_F \sim \left(\frac{|F|}{JL}\right)^{1/2}, \tag{88}$$

with $u_{\text{typ}}(\Delta)$ as the typical velocity of equilibrium fluctuations at the length scale Δ [see Eq. (76)], demonstrates that in the considered case both quantities are of the same order, and therefore, the approach based on using Eqs. (74) with $R \sim \Delta$ is indeed justified. This allows us to conclude that our instanton is created by fluctuations of the effective random potential whose amplitude is typical for their length scale. In such a situation the only reason why the probability of the instanton is small is that the signs of these typical fluctuations have to be same in all $L/\tau(\Delta)$ independent time intervals of the length $\tau(\Delta)$. This provides a qualitative explanation why the expression for the action can be reduced to a very simple form $S \sim L/\tau(\Delta)$.

At large values of $-F$ the range of the applicability of Eq. (82) is restricted by the constraint $\Delta(F) \gg a_\nu, a_D$, which is required for making replacement (81). In particular, when $d<2$ and $\xi \ll x_0$ (so that $a_\nu \sim a_D \sim x_0$) one can expect that at $\Delta(F) \sim x_0$, that is, at

$$-F \sim F_c \sim \frac{T^2 L}{J x_0^2}, \tag{89}$$

a crossover takes place from dependence (82) to dependence (46).

On the other hand, in situations when ξ (or d) is too large for dependence (46) to have any range of applicability, one could expect to have a direct crossover between dependences (57) and (82). However, the range of the applicability of Eq. (57) describing the far-left tail corresponds to $\Delta(F) \ll \min[a_\xi, a_D]$ and of Eq. (82) describing the universal regime to $\Delta(F) \gg \max[a_\xi, a_D]$. Since we expect that in a general situation the two length scales, a_ξ and a_D, are essentially different, we have to admit the existence in such a case of an intermediate region in F, where the form of the left tail of $P_L(F)$ cannot be established without further investigation.

As it has been already mentioned above, at $d=1$ the value of the exponent z is known exactly. Substitution of $z=3/2$ and $a_\nu \sim x_0$ into Eqs. (82) and (83) then reproduces an estimate for $S(F)$ which up to unknown numerical factor coincides with Eq. (38) for $S(F)$ in the far-left tail. This shows that for $d=1$ and $\xi \ll x_0$ the dependence of S on all parameters in the universal part of the left tail is exactly the same as in its nonuniversal part at $F_c \ll -F \ll F_\xi$. In this particular case at $-F \sim F_c$ only a numerical coefficient in dependence (82) can experience a crossover. For $d=1$ and $\xi \gg x_0$, substitution of Eq. (79) with $a_D \sim \xi$ into Eq. (84) reproduces an estimate for $F_*(L)$ which has been obtained by Nattermann and Renz[27] from scaling arguments complemented by the assumption that at low enough temperatures $F_*(L)$ has to be temperature independent and follows also from the replica-symmetry-breaking analysis of Ref. 28.

For $d \neq 1$ the value of the exponent η in the universal part of the left tail, $\eta_- = z/[2(z-1)]$, does not coincide with its value in the far-left tail, where it is given by $2-d/2$ [see Eq. (46)]. Note that expression (82) decreases with the increase in L as long as $\eta_- > 1$, that is, $1<z<2$. This means that in the universal part of the left tail the condition which is necessary for the possibility of having a macroscopic optimal fluctuation (whose size is much larger then ξ) is changed from $d<2$ to $1<z<2$. On the other hand, when the renormalization effects are taken into account, the condition $0<d<4$ required for having a minimum of S with respect to Δ (see Sec. IV B) is replaced by $4/3<z<2$. Thus, the range of the applicability of Eq. (82) is not restricted to $0<d<2$ (as in the case of the analogous expression for the far-left tail) but extends itself to the whole region of parameters where the strong-coupling phase does exist and $z>4/3$ (the condition $z<2$ always has to be fulfilled, see Sec. VI A). Note that for $d=1$ the value of $\zeta \equiv 1/z$ is equal to 2/3 and according to numerical simulations goes down with the increase in d.[3] This means that the restriction $z>4/3$ is fulfilled for any physical dimension.

C. Universal part of the right tail

One could expect the approach based on the application of the replacements (81) to be applicable also for the description of the universal part of the right tail. However, it turns out that in this case the situation is more complex. This can be understood by comparing the size of the optimal fluctuation $\Delta_+(F)$, given by Eq. (62), with the length scale $R_*(F)$ at which the typical velocity of equilibrium fluctuations $u_{\text{typ}}(R_*)$, given by Eq. (76), becomes comparable with

$$u_F \sim \frac{F}{J\Delta_+} \sim \left(\frac{F}{JL}\right)^{1/2}, \tag{90}$$

the characteristic velocity inside the optimal fluctuation with the size $\Delta_+(F)$. In Eq. (90) we have used the estimate for $\Delta_+(F)$ given by Eq. (62), which has led to exactly the same estimate for u_F in terms of $|F|$ as in the left tail [see Eq. (88)]. This means that in both tails $R_*(F)$ has to be of the same order. On the other hand, in Sec. VI B we have established that in the left tail the relation $u_F \sim u_{\text{typ}}(R_*)$ holds precisely when $R_* \sim \Delta(F)$. This allows one to conclude that in the right tail,

$$R_*(F) \sim \Delta(-F), \tag{91}$$

where $\Delta(F)$ is the instanton width in the left tail given by Eq. (80).

Accordingly, for the creation of the optimal fluctuation whose size $\Delta_+(F)$ is much larger than $R_*(F)$ (as it is required in the case of the right tail), the fluctuations of the effective random potential with length scale $\Delta_+(F)$ should have ampli-

tudes much larger than typical. Naturally, the probability of such fluctuations is strongly suppressed and cannot be estimated by using Eqs. (81).

The most effective way of formation of a fluctuation whose amplitude u_F substantially exceeds the typical velocity of fluctuations at the corresponding length scale consists in formation of a set of fluctuations with smaller length scales, such that for them the amplitudes of the order of u_F are typical. This means that the length scales of these fluctuations should be of the order of $R_*(F)$, and accordingly, the estimate for the action should include an additional factor $(\Delta_+/R_*)^d$ which takes into account the need for the spatial coherence of these fluctuations. This leads to

$$S(F) \sim \frac{L}{\tau(R_*)}\left(\frac{\Delta_+}{R_*}\right)^d \sim \left(\frac{F}{F_*}\right)^{\eta_+}, \tag{92}$$

where F_* is the same characteristic free-energy scale as in the universal part of the left tail [see Eq. (84)], whereas the exponent η_+ is given by

$$\eta_+ = \frac{(1+d)z}{2(z-1)}. \tag{93}$$

In terms of the renormalization approach exactly the same result is obtained if the renormalization is stopped not at the scale $\Delta_+(F)$, corresponding to the total size of the optimal fluctuation but at a smaller scale R_* (at which the fluctuations stop to be strong enough for inducing the renormalization), that is, by using Eq. (64) with the replacement

$$D \to D_{\mathrm{eff}}(R_*) \sim D\left(\frac{R_*}{a_D}\right)^{4+d-3z}, \tag{94}$$

where $R_* \sim \Delta(-F)$. Since the value of $D_{\mathrm{eff}}(R_*)$ does not depend on Δ_+, the condition for the existence of a minimum of S with respect to Δ_+ remains the same as has been found when deriving Eq. (64), $0<d<4$. On the other hand, in the universal part of the right tail the condition required for the possibility of having a macroscopic optimal fluctuation (whose size is much larger then ξ) is changed from $d<2$ to $1<z<2$, which in the strong-coupling phase anyway has to be fulfilled [see Sec. VI A]. Therefore, the range of the applicability of Eq. (92) is restricted from above not by $d=2$ (as in the case of the analogous expression for the far-right tail) but by $d=4$.

Note that in contrast to exponent η_- given by Eq. (83), exponent η_+ depends both on z and d. However, the ratio of these two exponents does not depend on z,

$$\frac{\eta_+}{\eta_-} = 1 + d, \tag{95}$$

and therefore is known exactly. The fact that in the regime where the renormalization effects are important both tails of the free-energy distribution function incorporate the same characteristic free-energy scale F_* confirms that this regime corresponds to studying the universal form of this distribution function.

A comparison of Eq. (62) with Eq. (80) allows one to verify that the condition $\Delta_+(F) \gg R_*(F)$, on which we have relied when deriving Eq. (92), is equivalent to $S(F) \gg 1$ and therefore is always satisfied as soon as we are dealing with the tail. Another condition whose fulfillment is required to justify replacement (94) is related to the quasistationarity of the problem. Namely, the total evolution time L has to be much larger than the characteristic relaxation time $\tau(R_*) \sim R_*^2/\nu_{\mathrm{eff}}(R_*)$ which can associated with the length scale $R_*(F)$. For $R_*(F) \sim \Delta(-F)$, this condition is also reduced to $S(F) \gg 1$.

From the side of large F the range of the applicability of Eq. (92) is restricted by the constraint $R_* \gg a_D$, whose fulfillment is also required for making replacement (94). In particular, at $d<2$ and $\xi \lesssim x_0$ (when $a_D \sim x_0$), the crossover between dependences (92) and (64) can be expected to occur at $F \sim F_c$, where F_c is given by the same expression [Eq. (89)] as in the left tail. On the other hand, at $d>2$ the crossover between dependences (92) and (66) has to take place while $R_*(F)$ is still much larger than ξ. In this situation we expect that the two contributions to $P_L(F)$ [one from the "macroscopic" instanton, corresponding to dependence (92) and the other from the "microscopic" instanton corresponding to dependence (66)] can coexist with each other and the crossover has to occur when they become comparable with each other.

VII. CONCLUSION

In the present work we have studied the form of the tails of the free-energy distribution function $P_L(F)$ in the directed polymer problem both for a δ-correlated random potential and for the case of a finite correlation length ξ. In all regimes that we have investigated the tails have a stretched-exponential form,

$$-\ln P_L(\pm F) \sim \left[\frac{F}{F_*(L)}\right]^{\eta_\pm}, \tag{96}$$

with $F_*(L) \propto L^{\omega_\pm}$ and therefore can be characterized by the two exponents whose values depend on the dimensionality of the space in which the polymer is imbedded. We use letter d to denote the transverse dimensionality of this space, that is, the number of components of the displacement vector $\mathbf{u}$.

For sufficiently large fluctuations of F the form of the tails of $P_L(F)$ is determined by the form of the most optimal fluctuation of a random potential which is sufficient for achieving a given value of F. For a δ-correlated random potential and $d<2$ the minimization of the action corresponding to such a fluctuation allows one to show that in the far-left tail,

$$\eta_- = \frac{4-d}{2}, \quad \omega_- = \frac{2-d}{4-d}. \tag{97}$$

The same values of η_- and ω_- have been obtained by Monthus and Garel[29] by constructing a generalization of the Imry-Ma scaling argument (based on a disorder-dependent Gaussian variational approach introduced in Ref. 30). However, the approach of Ref. 29 leaves one in doubt on what is the range of its applicability (and if such a range exists at all), whereas the methods used in this work allowed us to establish that the exponents (97) are applicable in the most distant part of the left tail corresponding to the nonuniversal regime.

At $d \geq 2$ the problem with strictly δ-functional correlations of a random potential becomes ill-defined, so it becomes necessary to introduce some regularization. The natural way of doing it consists in assuming that a random potential correlations are characterized by a finite correlation radius ξ. In the case of $\xi > 0$ one finds that in the most distant part of the left tail the size of the optimal fluctuation of a random potential has to be comparable with ξ and the values of the exponents become superuniversal, that is, not dependent on d,

$$\eta_- = 2, \quad \omega_- = 1/2. \tag{98}$$

For $d < 2$ and not too large ξ, one can expect to have a crossover from regime (98) to regime (97).

The application of the optimal-fluctuation approach to the analysis of the right tail shows that for $d < 2$ the most distant part of this tail is described by

$$\eta_+ = \frac{4+d}{2}, \quad \omega_+ = \frac{2-d}{4+d}. \tag{99}$$

In contrast to the case of the left tail, the form of the most distant part of the right tail is insensitive to whether ξ is zero or finite. On the other hand, for $d > 2$ the size of the optimal fluctuation again becomes determined by ξ, which leads to the change in the exponents to

$$\eta_+ = 3, \quad \omega_+ = 0. \tag{100}$$

Note that the value of ω_+ given by Eq. (99) corresponds to the value of the roughening exponent,

$$\zeta_F = \frac{3}{4+d}, \tag{101}$$

which is known as "Flory exponent"[31] and follows from simple scaling arguments of Refs. 31, as well as from the Gaussian variational calculation of Mezard and Parisi[32] incorporating a hierarchical replica-symmetry breaking. Our analysis has revealed that this scaling analysis (which insofar has been assumed to be of little relevance because it cannot reproduce the exactly known value of $\zeta = 2/3$ at $d=1$) in reality is applicable for the description of the most distant (nonuniversal) part of the right tail of $P_L(F)$. However, it still remains unclear whether the appearance of the same exponent in the variational calculation in Ref. 32 (based on the *maximization* of the variational free energy of a system with $L=\infty$) is not more than a coincidence.

If the parameters of the system correspond to the strong-coupling phase, the decrease in $|F|$ makes the optimal-fluctuation approach no longer directly applicable because the size of the optimal fluctuation becomes too large (or its amplitude becomes too small) to neglect the renormalization of the parameters of the system by fluctuations. In such a situation, a consistent inclusion of the renormalization effects into account allows one to express the exponents in terms of the roughening exponent $\zeta = 1/z$ describing the behavior of displacement fluctuations inside an infinite polymer [see Eq. (71)]. For universal parts of left and right tails, one obtains, respectively,

$$\eta_- = \frac{1}{2(1-\zeta)}, \quad \eta_+ = \frac{1+d}{2(1-\zeta)}. \tag{102}$$

Not unexpectedly, one finds that the value of ω is the same for both tails and is equal to $2\zeta - 1$, as it could be expected from the universality. Quite remarkably, the ratio $\eta_+ / \eta_- = 1 + d$ does not depend on ζ.

The value of ζ is known exactly only at $d=1$, where $\zeta = 2/3$. In this case the values of $\eta_- = 3/2$ and $\eta_+ = 3$ which follow from Eqs. (102) are in perfect agreement with the exact solution[11] of the polynuclear growth (PNG) model, which is accepted to belong to the same universality class. In terms of the directed polymer problem the PNG model corresponds to the Poisson distribution of identical pointlike impurities and a rather peculiar limit of vanishing elasticity, $J=0$, and zero temperature.[11,33] For this model the form of the distribution function $P_L(F)$ in the universal regime, as well as the scaling function $g(\alpha)$ entering Eq. (69), is known exactly.[11,34] The consistency between our results and that of Ref. 11 confirms that the directed polymer problem defined by Eq. (2) and the PNG model indeed belong to the same universality class.

The nonuniversal tails in the PNG model have been analyzed in Ref. 35. Naturally, in the nonuniversal regime even the models belonging to the same universality class can have different tails. The difference is especially evident in the case of what we call the far-right tail because in the PNG model the energy is by definition bounded from above and therefore its distribution function has to vanish for large enough positive fluctuations. On the other hand, it follows from Ref. 35 that in the PNG model the far-left tail is described by $S(F) \propto F \ln(-F/L)$ and, thus, also has nothing in common with the far-left tail of the model considered in this work.

In terms of the exponent $\omega = 2\zeta - 1$, Eqs. (102) can be rewritten as

$$\eta_- = \frac{1}{1-\omega}, \quad \eta_+ = \frac{1+d}{1-\omega}. \tag{103}$$

Our results demonstrate that in model (2) the analogous relations are fulfilled also in nonuniversal regimes (where ω is not obliged to coincide with $2\zeta - 1$ and be the same in both tails) as soon as the size of the optimal fluctuation is comparable with the total length of a string. For the far-left tail this has been known[3,36] from the Kardar-Zhang replica approach. Recently both relations (103) have been derived by Monthus and Garel[36] with the help of a recursive procedure for the zero-temperature problem on a hierarchical diamond lattice whose effective dimension is equal to d. These authors have also suggested that the same relations can be expected to hold on all hypercubic lattices. Although, in our opinion, the argument accompanying this proposal does not take into account some important differences between hypercubic and hierarchical lattices, the results derived in this work confirm its correctness.

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APPENDIX A: THE REPLICA APPROACH

The replica approach to the directed polymer problem is based on calculating the moments $Z_n \equiv \overline{Z^n}$ of the distribution of the partition function $Z \equiv z(L,0)$ and allows one to find the far-left tail of $P_L(F)$ without relying on the analytical continuation of n to 0. Kardar[4] was the first to notice that for any integer $n>1$ (and large enough polymer length L) Z_n with an exponential accuracy can be approximated as

$$Z_n \approx \exp[-E_0(n)L/T], \tag{A1}$$

where $E_0(n)$ is the ground-state energy of the quantum-mechanical Hamiltonian,

$$\hat{H}_n = -\frac{T^2}{2J}\sum_{a=1}^{n}\nabla_a^2 - \frac{1}{2T}\sum_{a=1}^{n}\sum_{b=1}^{n}U(\mathbf{x}_a-\mathbf{x}_b), \tag{A2}$$

describing n bosons whose mass is equal to J (with T playing the role of $\hbar$) and interaction to $-U(\mathbf{x})/T$. In a $(1+1)$-dimensional system with a δ-correlated random potential, $U(x)=U_0\delta(x)$, the ground-state wave function for the Hamiltonian (A2) and its energy,

$$E_0(n) = -\frac{U(0)}{T}n - \frac{JU_0^2}{24T^4}n(n^2-1), \tag{A3}$$

can be found exactly.[37] This gives

$$Z_n \propto \exp\left(\frac{JU_0^2}{24T^5}n^3L\right), \tag{A4}$$

where the linear in n term in $E_0(n)$ has been omitted because it can be eliminated by a constant shift of the potential $V(t,\mathbf{x})$ in Eq. (2). Note that the form of Eq. (A4) does not depend on the initial condition. The choice of the initial condition manifests itself only in the form of a prefactor which in the first approximation can be ignored.

Since $Z^n \equiv \exp(-nF/T)$, Z_n can also be expressed in terms of $P_L(F)$, the distribution function of the free energy $F \equiv f(L,0) = -T \ln Z$,

$$Z_n = \int_{-\infty}^{+\infty} dF P_L(F)\exp(-nF/T). \tag{A5}$$

It is clear that the integral in Eq. (A5) can be convergent for any integer $n>1$ only if the left tail of $P_L(F)$ decays faster than exponentially. In such a case the integration in Eq. (A5) can be performed with the help of the saddle-point approximation. A comparison of the two expressions for Z_n [Eqs. (A4) and (A5)] has led Zhang to conclusion[7] that the algebraic growth of $\ln Z_n \propto Ln^{1/\omega}$ at large n can be recovered only if one assumes that when $-F$ is large $S(F) \equiv -\ln[P_L(F)]$ is proportional to $(-F/L^{\omega})^{\eta}$ with $\eta = 1/(1-\omega)$.[3] Thus, $1/\omega = 3$ corresponds to $\eta = 3/2$.

It is not hard to check that expression (38) for $S(F)$ derived in Sec. IV A allows one to reproduce not only the power of n but also the full expression (A4), including the coefficient in front of n^3. For such a form of the tail the integral in Eq. (A5) at positive n is dominated by the vicinity of the saddle-point situated at

$$F = -F_c(L)n^2, \quad F_c(L) \equiv \frac{JU_0^2L}{8T^4}, \tag{A6}$$

where the full expression standing in the exponent,

$$\tilde{S}(F) = -S(F) - nF/T, \tag{A7}$$

has a maximum with respect to F. Substitution of Eq. (A6) into Eq. (A7) leads then to Eq. (A4), whereas calculation of $\partial^2\tilde{S}(F)/\partial F^2 = 1/(2TF_cn)$ allows one to verify that the condition for the applicability of the saddle-point approximation has a form $(F_c/T)n^3 \gg 1$ and is automatically fulfilled for any integer $n>1$ as soon as $F_c \gg T$, which anyway is required for the applicability of Eq. (A1).

Since the explicit expression for Z_n given by Eq. (A4) can be derived only at integer $n>1$, the region of the applicability of the replica approach is restricted to $-F \gg F_c(L)$, that is, coincides with the region where the left tail can be found with the help of the optimal-fluctuation approach (see Sec. IV A) without taking into account the renormalization effects. However, the analysis of Sec. VI B has demonstrated that in the $(1+1)$-dimensional systems with δ-functional correlations the form of the left tail in the region $F_* \ll -F \ll F_c$, where the renormalization effects have to be taken into account, remains qualitatively the same as for $-F \gg F_c$. In terms of the replica approach this means that the analytical continuation of the partition function of model (A2) even at $n<1$ has to behave itself as if it was dominated by the contribution from the same state as at $n>1$, although at $n<1$ this state no longer has the lowest energy.[6] The reasons for that still remain to be elucidated.

For $\xi>0$ the form of the ground state of the Hamiltonian (A2) cannot be found exactly even when $d=1$. However, for $\xi \gg x_0$ the value of $E_0(n)$ in a $(1+1)$-dimensional system with $\xi>0$ can be found rather accurately for any integer $n \geq 1$, because in this regime the typical distance between bosons is much smaller then the radius of their interaction[28] and thus the main contribution to $E_0(n)$ is given simply by $-U(0)n^2/2T$. A comparison of $Z_n \sim \exp[U(0)Ln^2/2T^2]$ with Eq. (A5) then immediately leads to the conclusion that the form of the far-left tail must be described by $S=F^2/2U(0)L$, as it has been already derived in Sec. IV C with the help of the optimal-fluctuation approach. From the origin of this result it is clear that it has to be applicable for the description of the most distant part of the left tail at finite ξ for any d. The two contradicting attempts of generalizing the replica approach to $d>1$ in the regime when the main contribution to $E_0(n)$ is determined by the full form of $U(\mathbf{x})$ (and therefore cannot be found without introducing some additional approximations) have been undertaken by Zhang[38] and Kolomeisky.[39]

APPENDIX B: FIXED INITIAL CONDITION

If at $t=0$ the polymer is fastened at $x=0$, the initial condition for the partition function $z(t,x)$ has to be written as

$$z(0,x) = x_d \delta(x), \tag{B1}$$

where a dimensional factor x_d (with the same dimensionality as x) has to be inserted to make $z(t,x)$ a dimensionless quantity. This factor has no physical meaning and must drop out from all physical quantities.

In the absence of pumping the initial condition (B1) leads to

$$z(t,x) = \frac{x_d}{(4\pi\nu t)^{1/2}} \exp\left[-\frac{x^2}{4\nu t}\right] \equiv z^{(0)}(t,x) \tag{B2}$$

and

$$f(t,x) = \frac{Jx^2}{2t} + \frac{T}{2}\ln\frac{4\pi\nu t}{x_d^2} \equiv f^{(0)}(t,x), \tag{B3}$$

which in terms of $u \equiv \nabla_x f(t,x)/J$ corresponds to

$$u(t,x) = x/t \equiv u^{(0)}(t,x). \tag{B4}$$

Accordingly, the initial condition for $f(t,x)$ can be formulated as

$$\lim_{t\to 0}[f(t,x) - f^{(0)}(t,x)] = 0. \tag{B5}$$

In such a situation it seems to be convenient to count off the free energy from its value in the absence of pumping introducing

$$\tilde{f}(t,x) = f(t,x) - f^{(0)}(t,x), \tag{B6}$$

(i) because $\tilde{f}(x,t)$ in contrast to $f(t,x)$ does not depend on x_d and (ii) because this allows to rewrite the initial condition (B5) as

$$\tilde{f}(0,x) = 0. \tag{B7}$$

Accordingly, in the case of fixed initial condition F should be redefined as

$$F = \tilde{f}(L,0) = f(L,0) - f^{(0)}(L,0). \tag{B8}$$

It is clear that the stationary solution (32) in no way resembles the initial condition (B5). However, like in the case of the free initial condition, it turns out possible to modify this solution *without increasing the action* in a way which eliminates the inconsistency with the initial condition.

This solution has to interpolate between the soliton at small x and the dependence $f(t,x)=f^{(0)}(t,x)$, which has to survive in the regions that in a given moment are too far from the soliton to feel its presence. One can expect that the time of the formation of a solution, t_1, will be of the same order as for free initial condition, $t_1 \sim \nu/v^2$, because at $t \gg t_1$ and $x \sim \Delta$ the velocity of the flow (B4) produced by fixed initial condition is much smaller then the velocity $v \sim \nu/\Delta$ produced by the soliton with the width Δ. Then at $t \gg t_1$ and $\Delta \ll x \ll vt$ f has to be of the form

$$f(t,x) = -\frac{Jv^2}{2}(t-t_1) + Jv|x| + \frac{T}{2}\ln\frac{4\pi\nu t_1}{x_d^2} \equiv f^{(s)}(t,x), \tag{B9}$$

where the constant has been chosen in such a way that the extrapolation to $t=t_1$ gives $f(t_1,0)=f^{(0)}(t_1,0)$.

A comparison of Eq. (B3) with Eq. (B9) shows that the crossover between these two dependences has to take place in the vicinity of $|x|=vt$. Since we assume that at $|x| \gg \Delta$ the potential is absent, this crossover has to be described by

$$f(t,x) = f^{(s)}(t,x) - T \ln z_1(t,|x| - vt), \tag{B10}$$

where the function $z_1(t,x)$ is the exact solution of the diffusion equation $z_t = \nu z_{xx}$ which at $t=t_1' = t_1 \exp[(Jv^2/2T)t_1] \sim t_1$ [that is, when $f^{(s)}(t,vt)$ coincides with $f^{(0)}(t,vt)$] smoothly interpolates between $z_1 \approx 1$ at $-x \gg \Delta$ and $z_1 \approx \exp[-Jx^2/2Tt_1']$ at $x \gg \Delta$. Therefore, the asymptotic behavior of $z_1(t,x)$ at large t can be found by assuming

$$z_1(t_1',x) = \theta(-x) + \theta(x)\exp[-Jx^2/2Tt_1'], \tag{B11}$$

where $\theta(x)=\frac{1}{2}[1+\mathrm{sign}(x)]$ is the steplike function. However, for establishing the relation between S and F the exact form of $z_1(t,x)$ is irrelevant. We only have to be sure that at $x \approx -vt$ this quantity is very close to 1, and this is satisfied as soon as $t \gg t_1'$.

The main difference with the case of free initial condition appears in the relation between Δ and F. Subtraction of Eq. (B3) from Eq. (B9) shows that for fixed initial condition Eq. (37) should be replaced by

$$-F = \frac{T^2}{2J\Delta^2}L + \frac{T}{2}\ln\frac{L}{t_1}. \tag{B12}$$

However, for $L \gg t_1$ the second term in the right-hand side of Eq. (37) is much smaller then the first one (which is of the order of TL/t_1) and therefore can be neglected.

This allows one to conclude that in the case of fixed initial condition the main contribution to the action has exactly the same form as for free initial condition. The derivation above can be easily generalized for the case of $d>1$, as soon as one can assume that the optimal fluctuation of a random potential remains almost uniform along t.

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